Estimating the error of linear approximations

Given the previous discussion, it should not come as too big a surprise that some information about the second derivative of the function will help us get an idea on the potential error our linear approximation is making.

A result

Given a differentiable function f for which we are approximating values around the point a on the x-axis using a linear approximation.

If the absolute value of the second derivative is at most a value M on some interval I around the point a, then the error made by using the linear approximation at a point x (that has to be in the interval I) is no more than

$$\text{error} \leq \frac{M}{2}(x-a)^2$$

for estimate of $\sqrt{9.12}$, notice $(9.12-9)^2$ versus $(9.12-4)^2$

An illustration of this result

Find M for estimate of $\sqrt{9.12}$ using $L_9(9.12)$

size of $f''(x)$ "near" 9.

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

really careful approach:

worry only about $9 \leq x \leq 9.12$

maximum $|f''(x)|$ over interval $[9, 9.12]$

here $|f''(x)| = |-\frac{1}{4}x^{-\frac{3}{2}}| = \frac{1}{4}|x^{-\frac{3}{2}}|$ for $x>0$, $x^{-\frac{3}{2}} > 0$

$h(x) = \frac{1}{4}x^{-\frac{3}{2}}$

$h'(x) = -\frac{3}{2}\cdot\frac{1}{4}\cdot x^{-\frac{5}{2}}$ only zero when $x=0$

near $x=9$ (for any $x>0$) $h'(x) < 0$

says that $h(x) = |f''(x)|$ is only decreasing

so max over $[9, 9.12]$ is at $x=9$

take $M = \max_{\text{over } [9,9.12]} |f''(x)| = |f''(9)| = \frac{1}{4}9^{-\frac{3}{2}} = \frac{1}{4}\cdot\frac{1}{3^3} = \frac{1}{4}\cdot\frac{1}{27}$

$$\text{error} \leq \frac{\left(\frac{1}{4}\cdot\frac{1}{27}\right)}{2}\cdot(9.12-9)^2$$